STABILITY OF MOTION DURING A FINITE TIME INTERVAL

(OB USTOICHIVOSTI DVIZHENIIA NA KONECHNOM INTERVALE VREMENI)

PMM Vol.23, No.2, 1959, pp. 230-238

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(Received 5 November 1958)

1. Statement of the Problem. In practice cases are encountered when it is required to know stability characteristics of physical systems not for the whole time interval $t \geqslant t_0$ (stability in Liapunov's sense) but for some finite interval of time $t_0 \leqslant t \leqslant T$.

We will call the unperturbed motion stable with respect to given ϵ and C in a finite time interval $t_0 \leqslant t \leqslant T$, if at $t = t_0$ the following holds good:

$$\sum_s x_{s0}^2 \leqslant \varepsilon \tag{1.1}$$

and for att t in the interval $t_0 \leqslant t \leqslant T$ the following relationship is satisfied:

$$\sum_s x_s^2 \leqslant C \tag{1.2}$$

Here T, ϵ, and C are given.

Let us find conditions for stability (in the above sense) of unperturbed motion of a system for a few cases.

2. Linear Systems with Variable Coefficients. In this case the equations of the perturbed motion of the system have the following form:

$$\frac{dx_s}{dt} = p_{s1}(t)\,x_1 + \ldots + p_{sn}(t)\,x_n \qquad (s = 1, \ldots, n) \tag{2.1}$$

where $p_{sr}(t)$ are real bounded continuous functions of time t and may depend on some parameters.

To solve the problem let us consider the function

$$2V = e^{-\alpha t}(x_1^2 + \ldots + x_n^2) \tag{2.2}$$

where α is a positive number which we are leaving undefined for the time being. This function was first used by Liapunov in his proof of the theorem of boundedness of characteristic numbers.

In virtue of (2.1) we have

$$\frac{dV}{dt}=\frac{\partial V}{\partial t}+\sum_s\frac{\partial V}{\partial x_s}\frac{dx_s}{dt}=-\frac{\alpha}{2}e^{-\alpha t}(x_1^2+\dots+x_n^2)+e^{-\alpha t}\sum_s x_s\frac{dx_s}{dt}=e^{-\alpha t}W \quad (2.3)$$

Here

$$W=\sum_{sr}\left(\frac{p_{sr}+p_{rs}}{2}-\delta_{sr}\frac{\alpha}{2}\right)x_sx_r \qquad \delta_{sr}=\begin{cases}1 & \text{for } s=r\\ 0 & \text{for } s\neq r\end{cases}$$

Let us choose α and the parameters of the coefficients p_{sr} such that the quadratic form W would be negative definite. In accordance with Sylvester's theorems it is sufficient and necessary to this end to have the inequalities

$$D_r=\begin{vmatrix}-p_{11}+\frac{\alpha}{2} & -\frac{p_{12}+p_{21}}{2} & \dots & -\frac{p_{1r}+p_{r1}}{2}\\ \dots & \dots & \dots & \dots\\ -\frac{p_{r1}+p_{1r}}{2} & -\frac{p_{r2}+p_{2r}}{2} & \dots & -p_{rr}+\frac{\alpha}{2}\end{vmatrix}>0 \qquad (r=1,\dots,n) \quad (2.4)$$

Under these conditions $-W$ will be positive definite quadratic form and, therefore, it is always possible to find a positive number μ such that the following will hold good:

$$-W=-\sum_{s,r}\left(\frac{p_{sr}+p_{rs}}{2}-\delta_{sr}\frac{\alpha}{2}\right)x_sx_r>\frac{\mu}{2}(x_1^2+\dots+x_n^2) \qquad (2.5)$$

Substituting inequality (2.5) into (2.3), by virtue of (2.2), we obtain

$$\frac{dV}{dt}<-\mu\frac{1}{2}e^{-\alpha t}(x_1^2+\dots+x_n^2)=-\mu V \qquad (2.6)$$

Let us assume that at $t=t_0$ the point $(x_{10},\dots,x_{n0})$ lies on the sphere (ϵ), i.e. $x_{10}^2+\dots+x_{n0}^2=\epsilon$ and that at some instant t the point reaches the sphere (C), i.e. $x_1^2+\dots+x_n^2=C$.

Let us find this instant. Let $t_0=0$. Then, integrating inequality (2.6), we obtain

$$V<V_0e^{-\mu t}$$

But, according to (2.2)

$$V_0=\frac{1}{2}\varepsilon, \qquad V=\frac{1}{2}e^{-\alpha t}C.$$

Therefore

$$\frac{C}{\varepsilon} < e^{(\alpha-\mu)t} \qquad \text{or} \qquad t > \frac{1}{\alpha-\mu}\ln\frac{C}{\varepsilon} \tag{2.7}$$

From this we can see that if the last inequality has the opposite sign then the point $(x_1, \ldots, x_n)$ cannot reach the sphere (C). Therefore, if

$$T = \frac{1}{\alpha-\mu}\ln\frac{C}{\varepsilon} \tag{2.8}$$

then for all $t \leqslant T$ no trajectory can extend beyond the sphere (C). Further, it may be seen from (2.7) that the smaller the sphere (ϵ), the greater the time t. Therefore, if conditions (2.8) are satisfied then inequalities (1.1) and (1.2) will also be satisfied.

Equation (2.8) contains a and μ, which must be so chosen that equation (2.5) is satisfied. Let us write (2.5) in the following form:

$$W_1 = \sum_{s,\, r}\left[-\frac{p_{sr}+p_{rs}}{2} + \delta_{sr}\left(\frac{\alpha}{2}-\frac{\mu}{2}\right)\right]x_s x_r > 0 \tag{2.9}$$

Thus w_1 is a positive definite form.

Let

$$\lambda = \alpha - \mu \tag{2.10}$$

The necessary and sufficient conditions for the quadratic form (2.9) to be positive are of the form

$$D_r = \begin{vmatrix} -p_{11}+\frac{\lambda}{2} & -\frac{p_{12}+p_{21}}{2} & \ldots & -\frac{p_{1r}+p_{r1}}{2} \\ \cdot & \cdot & \cdots & \cdot \\ -\frac{p_{r1}+p_{1r}}{2} & -\frac{p_{r2}+p_{2r}}{2} & \ldots & -p_{rr}+\frac{\lambda}{2} \end{vmatrix} > 0 \qquad r=1, \ldots, n) \tag{2.11}$$

From the above considerations the following conclusion for system (2.1) can be stated.

Theorem 1. The unperturbed motion will be stable with respect to given ϵ and C in the finite interval of time $t_0 \leqslant t \leqslant T$ if conditions (2.11) are satisfied.

3. A Special Case of a Linear System and a Linear System with Constant Coefficients. Let us consider a special case when for system (2.1) in the time interval $t_0 \leqslant t \leqslant T$ the following equations hold good:

$$p_{sr}(t) = c_{sr} + \delta f_{sr}(t) \tag{3.1}$$

where c_{sr} are constants, δ is a sufficiently small number, and $f_{sr}(t)$ are bounded functions. In this case system (2.1) has the form

$$\frac{dx_s}{dt} = c_{s1}x + \ldots + c_{sn}x_n + \delta f_{s1}x_1 + \ldots + \delta f_{sn}x_n \quad (s = 1, \ldots, n)$$

The total derivative of function V (2.2) by virtue of this system has the form

$$\frac{dV}{dt} = e^{-\alpha t}\left[\sum_{s,r}\left(\frac{c_{sr}+c_{rs}}{2} - \delta_{sr}\frac{\alpha}{2}\right)x_s x_r + \sum_{s,r}\delta\frac{f_{sr}+f_{rs}}{2}x_s x_r\right]$$

The coefficients of the second quadratic form in brackets on the right-hand side of this equation are sufficiently small for the sign of the function in brackets to be completely determined by the sign of the first quadratic form.

Performing calculations analogous to those of the preceding section we obtain stability conditions of form (2.11), where all $p_{sr}(t)$ are replaced by constants c_{sr}.

Let us now consider a linear function with constant coefficients:

$$\frac{dx_s}{dt} = c_{s1}x_1 + \ldots + c_{sn}x_n \qquad (s = 1, \ldots, n) \tag{3.2}$$

The total derivative of the function V (2.2) by virtue of (3.2) has the form

$$\frac{dV}{dt} = We^{-\alpha t}, \quad W = \sum_{s,r}\left(\frac{c_{sr}+c_{rs}}{2} - \delta_{sr}\frac{\alpha}{2}\right)x_s x_r$$

To make $dV/dt < 0$, the following conditions must be satisfied:

$$D_r = \begin{vmatrix} -c_{11}+\frac{\alpha}{2} & -\frac{c_{12}+c_{21}}{2} & \ldots & -\frac{c_{1r}+c_{r1}}{2} \\ \ldots & \ldots & \ldots & \ldots \\ -\frac{c_{r1}+c_{1r}}{2} & -\frac{c_{r2}+c_{2r}}{2} & \ldots & -c_{rr}+\frac{\alpha}{2} \end{vmatrix} > 0 \qquad (r = 1, \ldots, n) \tag{3.3}$$

Under these conditions - W will be a positive definite symmetric quadratic form. Its extremum on the sphere $x_1^2 + \ldots + x_n^2 = A$ is defined by the expression

$$-W = -\sum_{s,r}\left(\frac{c_{sr}+c_{rs}}{2} - \delta_{sr}\frac{\alpha}{2}\right)x_s x_r = \frac{\varkappa}{2}\sum_s x_s^2 \tag{3.4}$$

where $1/2\ \kappa$ are the roots of the secular equation

$$X(\varkappa)=\left|-\frac{c_{sr}+c_{rs}}{2}+\delta_{sr}\left(\frac{\alpha}{2}-\frac{\varkappa}{2}\right)\right|=0 \tag{3.5}$$

According to Sylvester's theorem all these roots will be real and positive, since the quadratic form is positive definite.

According to (3.4) we have:

$$\frac{dV}{dt}=-\varkappa\frac{1}{2}e^{-\alpha t}(x_1^2+\ldots+x_n^2)=-\varkappa V$$

Carrying out a calculation analogous to that of the preceding section, we obtain an equality for the time of arrival of the point on the sphere (C):

$$t=\frac{1}{\alpha-\varkappa}\ln\frac{C}{\varepsilon}$$

Let $a-\kappa=\lambda$. It is seen that λ will be roots of the equation

$$\Delta(\lambda)=\left|-\frac{c_{sr}+c_{rs}}{2}+\delta_{sr}\frac{\lambda}{2}\right|=0 \tag{3.6}$$

All the roots of this secular equation will also be real.

Let us examine only positive roots of equation (3.6). Let these be $\lambda_1, \ldots, \lambda_k$, and let $0<\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_k$.

Then on the basis of the equality $t=\lambda^{-1}\ln(C/\epsilon)$, the greatest possible travel time of a point from the sphere (ϵ) to the sphere (C) is equal to $\lambda_1^{-1}\ln(C/\epsilon)$, and the least possible travel time is equal to $\lambda_k^{-1}\ln(C/\epsilon)$. If we take

$$T\leqslant\frac{1}{\lambda_k}\ln\frac{C}{\varepsilon} \tag{3.7}$$

then conditions (1.1) and (1.2) will be satisfied. Therefore, we have the following theorem.

Theorem 2. For the unperturbed motion of the system to be stable with respect to given ϵ and C in a finite time interval $t_0\leqslant t\leqslant T$, it is sufficient for conditions (3.7) to be satisfied.

4. Nonlinear system. Let us consider a more general case. Let

$$\frac{dx_s}{dt}=p_{s1}(t)x_1+\ldots+p_{sn}(t)x_n+X_s \qquad (s=1,\ldots,n) \tag{4.1}$$

where $X_s=X_s(t, x_1, \ldots, x_n)$ are holomorphic functions of the variables $x_1, \ldots, x_n$ and first terms of their expansions are not lower than of the second order. Coefficients of these functions are real continuous bounded functions of t.

1. ϵ and C are sufficiently small. The total derivative of the function (2.2) in virtue of (4.1) has the form

$$\frac{dV}{dt} = e^{-\alpha t}\Big[W + \sum_s x_s X_s\Big] \qquad \Big(W = \sum_{s,r}\Big(\frac{p_{sr}+p_{rs}}{2} - \delta_{sr}\frac{\alpha}{2}\Big)x_s x_r\Big) \quad (4.2)$$

In this case the sign of dV/dt is completely determined by the sign of W, so that the inequality $dV/dt < 0$ is satisfied for conditions (2.4).

Satisfying (2.4), we have

$$\frac{dV}{dt} < -\mu\, \frac{1}{2} e^{-\alpha t}(x_1^2 + \ldots + x_n^2)$$

As in Section 2, we obtain

$$t > \frac{1}{\alpha - \mu}\ln\frac{C}{\varepsilon}$$

Let

$$T = \frac{1}{\lambda}\ln\frac{C}{\varepsilon} \qquad (\lambda = \alpha - \mu)$$

Then (1.1) and (1.2) will be satisfied. Or, let us proceed this way: let

$$\sum_{s,r}\Big(\frac{p_{sr}+p_{rs}}{2} - \delta_{sr}\frac{\alpha}{2}\Big)x_s x_r + \sum_s x_s X_s < -\frac{\mu}{2}(x_1^2 + \ldots + x_n^2) \quad (4.3)$$

Introducing [3] the notation $S = \sum_s x_s X_s$, we have

$$|S| \leqslant R(t)(x_1^2 + \ldots + x_n^2)$$

where $R(t)$ is a positive function which is the upper exact limit of the function

$$\frac{1}{x_1^2 + \ldots + x_n^2}\Big|\sum_s x_s X_s\Big| \quad \text{for } t_0 \leqslant t \leqslant T$$

Let

$$\sum_{s,r}\Big(\frac{p_{sr}+p_{rs}}{2} - \delta_{sr}\frac{\alpha}{2}\Big)x_s x_r + R(t)(x_1^2 + \ldots x_n^2) < -\frac{\mu}{2}(x_1^2 + \ldots + x_n^2) \quad (4.4)$$

Obviously, if this condition is satisfied, then condition (4.3) applies Let us write (4.4) in different form:

$$W_1 = \sum_{s,r}\Big[-\frac{p_{sr}+p_{rs}}{2} + \delta_{sr}\Big(\frac{\lambda}{2} - R(t)\Big)\Big]x_s x_r > 0$$

For W_1 to be a positive form it is necessary and sufficient to satisfy the following inequalities:

$$D_r = \begin{vmatrix} -p_{11} + \frac{\lambda}{2} - R(t) & -\frac{p_{12}+p_{21}}{2} & \dots & -\frac{p_{1r}+p_{r1}}{2} \\ \dots & \dots & \dots & \dots \\ -\frac{p_{r1}+p_{1r}}{2} & -\frac{p_{r2}+p_{2r}}{2} & \dots & -p_{rr} + \frac{\lambda}{2} - R(t) \end{vmatrix} > 0$$

$$(r = 1, \dots n) \qquad (4.5)$$

Therefore, we have the following theorem for (4.1).

Theorem 3. For the unperturbed motion of system (4.1) to be stable with respect to given ϵ and C in a finite time interval $t_0 \leqslant t \leqslant T$, it is sufficient for conditions (4.5) to be satisfied.

Conditions (4.5) contain $R(t)$. We can obtain other sufficient but simpler conditions without $R(t)$.

Indeed, from the inequality (4.3) it follows that in case 1 the sign of the function on the left-hand side of this inequality is completely determined by the quadratic form. Thus for (4.3) to apply it is sufficient to satisfy the inequalities (2.11).

2. Case when ϵ and C are finite. In this case

$$\frac{dV}{dt} = e^{-\alpha t}\left[W + \sum_s x_s X_s\right] \qquad \left(W = \sum_{s,\,r}\left(\frac{p_{sr}+p_{rs}}{2} - \delta_{sr}\frac{\alpha}{2}\right)x_s x_r\right) \qquad (4.6)$$

but ϵ and C are finite and, therefore, the sign of dV/dt is not determined by the sign of the quadratic form W. Let

$$S = \sum_s x_s X_s$$

After carrying out computations analogous to those performed for case 1, we obtain conditions of form (4.5) and a theorem analogous to theorem 3.

It should be noted that in this case ϵ and C must be verified by conditions (4.5), which contain $R(t)$ and permit only a limited range over which the variables x_s may be varied.

Besides, in this case (ϵ and C are finite) conditions (2.11) do not apply.

3. Case when $p_{sr}(t) = c_{sr} + \delta f_{sr}(t)$, i.e. (3.1). In this case, in conformity with (4.1), we have

$$\frac{dx_s}{dt} = c_{s1}x_1 + \dots + c_{sn}x_n + \delta(f_{s1}x_1 + \dots + f_{sn}x_n) + X_s \qquad (s = 1, \dots, n) \qquad (4.7)$$

The total derivative of the function V (2.2) in virtue of (4.7) has the form

$$\frac{dV}{dt} = e^{-\alpha t}\Big[\sum_{s,r}\Big(\frac{c_{sr}+c_{rs}}{2} - \delta_{sr}\frac{\alpha}{2}\Big)x_s x_r + \sum_s x_s X_s + \sum_{s,r}\delta\frac{f_{sr}+f_{rs}}{2}x_s x_r\Big] \qquad (4.8)$$

If ϵ and C are sufficiently small, the sign of the function in brackets on the right-hand side of the equation is completely determined by the sign of the first quadratic form. In that case we obtain stability conditions of form (2.11) with all $p_{sr}(t)$ replaced by constants c_{sr}.

When ϵ and C are finite, the sign of the function in brackets is completely determined by the signs of the first two terms of this function. In that case we obtain stability conditions of form (4.5) with all $p_{sr}(t)$ replaced by constants c_{sr}.

5. Nonlinear System with steady Disturbances. Let us consider a system

$$\frac{dx_s}{dt} = p_{s1}(t)x_1 + \ldots + p_{sn}(t)x_n + X_s + R_s \qquad (s=1,\ldots n) \qquad (5.1)$$

where R_s describe the steady disturbances. For R_s we assume (in conformity with [3]) that R_s $R_s(t, x_1, \ldots, x_n)$ are real continuous bounded functions of t and x_s and that $R_s(t, 0, \ldots, 0) = 0$; and further, that

$$R_s(t, x_1, \ldots, x_n) = l_{s1}(t)x_1 + \ldots + l_{sn}(t)x_n + \Delta X_s \qquad (5.2)$$

where $l_{sr}(t)$ and ΔX_s have the same characteristics as the functions $p_{sr}(t)$ and X_s. In general R_s are unknown functions, but in many cases their values can be estimate.

According to (5.2) we have

$$\frac{dx_s}{dt} = (p_{s1}+l_{s1})x_1 + \ldots + (p_{sn}+l_{sn})x_n + X_s + \Delta X_s \qquad (s=1,\ldots,n) \qquad (5.3)$$

1) Let us consider a case when ϵ and C are sufficiently small.

The total derivative of the function V (2.2) in virtue of (5.3) has the form

$$\frac{dV}{dt} = e^{-\alpha t}\Big[\sum_{s,r}\Big(\frac{(p_{sr}+l_{sr})+(p_{rs}+l_{rs})}{2} - \delta_{sr}\frac{\alpha}{2}\Big)x_s x_r + \sum_s x_s(X_s + \Delta X_s)\Big]$$

In this case $dV/dt < 0$, if

$$D_r = \begin{vmatrix} -(p_{11}+l_{11})+\frac{\alpha}{2} & -\frac{(p_{12}+l_{12})+(p_{21}+l_{21})}{2} & \cdots & \frac{(p_{1r}+l_{1r})+(p_{r1}+l_{r1})}{2} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{(p_{r1}+l_{r1})+(p_{1r}+l_{1r})}{2} & -\frac{(p_{r2}+l_{r2})+(p_{2r}+l_{2r})}{2} & \cdots & -(p_{rr}+l_{rr})+\frac{\alpha}{2} \end{vmatrix} > 0$$

$$(r = 1, \ldots, n) \qquad (5.4)$$

We will state the results for system (5.1) without computations.

Theorem 4. For the unperturbed motion of system (5.1) to be stable with respect to given ϵ and C in the finite time interval $t_0 \leqslant t \leqslant T$, it is sufficient for the following inequalities to be satisfied:

$$D_r = \begin{vmatrix} -(p_{11}+l_{11})+\frac{\lambda}{2}-R(t) & \tau_{12} & \cdots & \tau_{1r} \\ \cdots & \cdots & \cdots & \cdots \\ \tau_{r1} & \tau_{r2} & \cdots & (p_{rr}+l_{rr})+\frac{\lambda}{2}-K(t) \end{vmatrix} > 0$$

$$(r = 1, \ldots, n) \qquad (5.5)$$

Here

$$\tau^{\lambda\mu} = \frac{(p_{\lambda\mu}+l_{\mu\lambda})+(p_{\mu\lambda}+l_{\mu\lambda})}{2}$$

$$S = \sum_s x_s(X_s+\Delta X_s), \qquad |S| \leqslant R(t)(x_1^2+\ldots+x_n^2)$$

Function $R(t)$ is positive and is the upper exact limit of the function

$$\frac{1}{x_1^2+\ldots+x_n^2}\left|\sum_s x_s(X_s+\Delta X_s)\right|$$

In addition to (5.5), the following sufficient conditions also apply:

$$D_r = \begin{vmatrix} -(p_{11}+l_{11})+\frac{\lambda}{2} & \tau_{12} & \cdots & \tau_{1r} \\ \cdots & \cdots & \cdots & \cdots \\ \tau_{r1} & \tau_{r2} & \cdots & (p_{rr}+l_{rr})+\frac{\lambda}{2} \end{vmatrix} > 0$$

$$(r = 1, \ldots, n) \qquad (5.6)$$

2. If ϵ and C are finite, then Theorem 4 and conditions (5.5) apply, while conditions (5.6) do not.

3. An investigation of the case when $p_{sr}(t) = c_{sr} + \delta f_{sr}(t)$, and $l_{sr}(t) = C'_{sr}$ = constant, may be carried out in the same way as in

Section 4; the results will be analogous to those of Section 4.

6. **Examples.** 1. Let us consider a second-order system:

$$\frac{dx_1}{dt} = p_{11}(t)\,x_1 + p_{12}(t)\,x_2, \qquad \frac{dx_2}{dt} = p_{21}(t)\,x_1 + p_{22}(t)\,x_2 \tag{6.1}$$

According to Theorem 1 and (2.11), the necessary conditions will be

$$-p_{11}(t) + \frac{\lambda}{2} > 0, \qquad \begin{vmatrix} -p_{11}(t) + \frac{\lambda}{2} & -\frac{p_{12}(t) + p_{21}(t)}{2} \\ -\frac{p_{21}(t) + p_{12}(t)}{2} & -p_{22}(t) + \frac{\lambda}{2} \end{vmatrix} > 0$$

Or, using (2.8) through (2.10), we have

$$p_{11}(t) + \frac{1}{2T}\ln\frac{C}{\varepsilon} > 0$$

$$\left(-p_{11}(t) + \frac{1}{2T}\ln\frac{C}{\varepsilon}\right)\left(-p_{22}(t) + \frac{1}{2T}\ln\frac{C}{\varepsilon}\right) - \frac{(p_{12}(t) + p_{21}(t))^2}{4} > 0 \tag{6.2}$$

If $p_{sr}(t)$ contain parameters, then they must be so chosen that the inequalities (6.2) are satisfied.

If $p_{sr}(t) = c_{sr} + \delta f_{sr}(t)$ then, according to the considerations of Section 3, the following conditions hold good:

$$-c_{11} + \frac{1}{2T}\ln\frac{C}{\varepsilon} > 0$$

$$\left(-c_{11} + \frac{1}{2T}\ln\frac{C}{\varepsilon}\right)\left(-c_{22} + \frac{1}{2T}\ln\frac{C}{\varepsilon}\right) - \frac{(c_{12} + c_{21})^2}{4} > 0 \tag{6.3}$$

2. Given a system

$$\frac{dx_1}{dt} = p_{11}(t)\,x_1 + p_{12}(t)\,x_2 + X_1, \qquad \frac{dx_2}{dt} = p_{21}(t)\,x_1 + p_{22}(t)\,x_2 + X_2 \tag{6.4}$$

According to (4.5), we have

$$p_{11} + \frac{\lambda}{2} - R(t) > 0 \qquad \left(\lambda = \frac{1}{T}\ln\frac{C}{\varepsilon}\right)$$

$$\left(-p_{11} + \frac{\lambda}{2} - R(t)\right)\left(-p_{22} + \frac{\lambda}{2} - R(t)\right) - \frac{(p_{12} + p_{21})^2}{4} > 0 \tag{6.5}$$

Here $R(t)$ is the upper exact limit of the function

$$\frac{|x_1X_1 + x_2X_2|}{x_1^2 + x_2^2}$$

Comparison of (6.5) and (6.2) reveals that stability conditions for system (6.4) are narrower than for the linear system. The larger $R(t)$, the narrower is (6.5). Thus, for $R(t) = 1/2\,\lambda$, we have

$$-p_{11} > 0, \; (-p_{11})(-p_{22}) - \frac{(p_{12}+p_{21})^2}{4} > 0 \tag{6.6}$$

These conditions are analogous to conditions (6.2) for linear system (6.1) when

$$\frac{1}{2T}\ln\frac{C}{\varepsilon} = 0$$

In this case $\epsilon = C$, and T is arbitrary.

When ϵ and C are sufficiently small, the $R(t)$ is also very small. In this case, in conformity with (2.11), we have

$$\begin{gathered} -p_{11} + \frac{1}{2T}\ln\frac{C}{\varepsilon} > 0 \\ \left(-p_{11} + \frac{1}{2T}\ln\frac{C}{\varepsilon}\right)\left(-p_{22} + \frac{1}{2T}\ln\frac{C}{\varepsilon}\right) - \frac{(p_{12}+p_{21})^2}{4} > 0 \end{gathered} \tag{6.7}$$

This is identical with (6.2) derived for the linear system.

3. For a system with steady disturbances

$$\begin{aligned} \frac{dx_1}{dt} &= p_{11}(t)x_1 + p_{12}(t)x_2 + X_1 + R_1 \\ \frac{dx_2}{dt} &= p_{21}(t)x_1 + p_{22}(t)x_2 + X_2 + R_2 \end{aligned} \tag{6.8}$$

and with conditions (5.2), we have (according to (5.5) and (5.6)) the following stability conditions:

$$-(p_{11}+l_{11}) + \frac{1}{2T}\ln\frac{C}{\varepsilon} - R(t) > 0$$

$$\left[-(p_{11}+l_{11}) + \frac{1}{2T}\ln\frac{C}{\varepsilon} - R(t)\right]\left[-(p_{22}+l_{22}) + \frac{1}{2T}\ln\frac{C}{\varepsilon} - R(t)\right] - \frac{[(p_{12}+l_{12})+(p_{21}+l_{21})]^2}{4} > 0 \tag{6.9}$$

and for sufficiently small ϵ and C we have

$$-(p_{11}+l_{11}) + \frac{1}{2T}\ln\frac{C}{\varepsilon} > 0 \tag{6.10}$$

$$\left[-(p_{11}+l_{11}) + \frac{1}{2T}\ln\frac{C}{\varepsilon}\right]\left[-(p_{22}+l_{22}) + \frac{1}{2T}\ln\frac{C}{\varepsilon}\right] - \frac{[(p_{12}+l_{12})+(p_{21}+l_{21})]^2}{4} > 0$$

In conclusion, the author expresses his deep gratitude to N.G. Chetaev for the statement of the problem and valuable comments.

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Translated by P.N.B.